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Searching for Nuclei Exhibiting the Critical Point Symmetry X(4)

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ABSTRACT

Through the application of the geometrical collective model, a number of nuclei have been subjected to an exhaustive investigation. A complete investigation has been conducted on the nuclei, which have a range of 2.60 ≤ R $_{4/2}$ ≤ 2.80. A comparative investigation was carried out in order to make a determination regarding the characteristics of the low-lying collective structure. A comparison was made between the energy levels that were measured for each individual nucleus and three distinct sets of theoretical calculations that were derived from the X(3), X(4), and X (5) models. For the purpose of this analysis, a comparison was made between the theoretical calculations and the experimentally measured energy levels. The nucleus that exhibits features that are consistent with the X(4) model is the target that is being sought after during the detection operation. For the entirety of this study, a certain set of criteria has been developed and put through a comprehensive analysis. There was a limited number of nuclei that were able to fulfill these requirements entirely.

1. INTRODUCTION

The atomic nucleus can be referred to as a quantal system with strongly interacting constituents in a very small volume with collective features [1]. Over the last two decades, a large amount of experimental data has been collected [2]. Several models try to explain it [3]. Among these are the geometric collective model (GCM) [4] proposed by Bohr and Mottelson, and the Interacting Boson Approximation (IBA) model [5]. Since then, the search for nuclei suitable for collective description has never ceased [4]. Nuclear collectivity is often classified in terms of three basic models: the spherical vibrator [6], the axially symmetric rotor [7] and the γ - soft rotor [8]. These are fundamental limits codified in the framework of the (IBA) model [5, 9] in terms of the $U(5), SU(3)$ and $O(6)$ dynamical symmetries, respectively. IBA model calculation of energy levels leads to values of $R_{4/2} = E(4_1^+)/E(2_1^+)$, this is the ratio of the excitation energy of the first 4^+ and first 2^+ excited states, equal to 2.00, 3.33, and 2.50 for the dynamical symmetries $U(5)$, $SU(3)$, and $O(6)$, respectively [10].

New solvable, parameter-free models with an overall scaling factor were proposed by Iachello at the beginning of this century [11, 12]. These describes the critical point symmetries (CPSs) at the end of one of the fundamental dynamical symmetries and the beginning of another one, where the potential should be flat [13] in terms of the zeros of Bessel function. The first model was the $E(5)$ (CPS) [11], which describes nuclei undergoing a second-order phase transition from the vibrational $U(5)$ shapes to γ - unstable $O(6)$ deformed shapes with $R_{4/2} = 2.2$. He carried out an exact separation of variables using potential, which depended only on β variable in the form of infinite square well potential [11]. After that, he proposed the $X(5)$ (CPS) which describes a first-order phase transition between the vibrational $U(5)$ shapes and the axially deformed rotor $SU(3)$ with $R_{4/2} = 2.90$ [12]. The solution of the two (CPSs) was based on an infinite square well potential in β. The potential of $E(5)$ (CPS) depended only on β, $u(\beta)$, while that of $X(5)$ separated into two parts, $u(\beta) + v(\gamma)$ and an approximate separation of variables could be possible with a value of $\gamma \simeq 0^o$.

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Bonatsos et al. inspired the $X(5)$ solution [12], they proposed a new (CPS), $Z(5)$ [14], for the phase shape transition from prolate to oblate with harmonic oscillator potential which has a minimum at $\gamma \simeq 30^o$ and typical infinite square well potential in β with a value of $R_{4/2} = 2.35$. In all the previous cases of the (CPSs) [11, 12, 14], the authors consider the usual Bohr Hamiltonian with five degrees of freedom, three Euler angles, and the two collective coordinates β and γ [7]. A rigid version of Z(5) model called Z(4) [15] was obtained when the authors fixed the value of the minimum of harmonic oscillator potential at $\gamma = 30^o$, the number of variables reduced and exact separation of variables and solution for the Davydov and Chaban Hamiltonian [16] with four variables could be possible [15]. Again, the separation of variables yields an infinite square well potential in β with solution obtained in terms of zeros of Bessel function, and the γ part leads to an equation solved by Meyer-ter-Vehn [17]. The resulting $Z(4)$ model can describe the (CPS) for spherical vibrator to rigid triaxial rotor [15] with $R_{4/2} = 2.226$ which is very close to that of $E(5)$ [11]. This similarity between the two (CPSs) is obvious in the similarity of the eigenstates in the ground state band, the β_1 band, the B(E2) values, and the odd levels in γ_1 band. The main variance appears in the even levels in the γ_1 band [15].

Using the same approach as $Z(4)$ [15], Bonatsos et al. derive a γ - rigid model version from $X(5)$ [12] (CPS) by fixing the γ variable to zero value, i.e $\gamma = 0^o$ exactly, according to the axially symmetric rotor. This will reduce the number of dimensions to three and lead to an $X(3)$ exactly separable and solvable model that uses potential depending only on β variable with $R_{4/2} = 2.44$ [13, 18, 19]. Budaca et al. established a new (CPS) by equally mixing the Hamiltonian of the γ - rigid $X(3)$ model [18, 19] and γ - stable $X(5)$ [12](CPS) . The resulting (CPS) is called $X(4)$ model [20] and can describe the shape phase transition from spherical to axially symmetric shapes with $R_{4/2} = 2.71$ [21]. The authors [20] found the spectrum of $X(4)$ (CPS) by using the same technique as $X(5)$, and using an approximate separation for the angular variables β and γ [12, 20].

The goal of this work is to locate nuclei that display the $X(4)$ model by conducting comprehensive research on candidate nuclei in the interval $2.60 \le R_{4/2} \le 2.80$ throughout the complete nuclear landscape. The judgment was made using specific criteria (see Section 3).

2. THEORETICAL CONCEPTS

In this section we will study only the three $X(n)$ models, $(n = 3,4,5)$ [12, 18, 20] briefly. We shall attempt to derive the spectral equations for the ground state and β_1 bands. In general, a combined axially symmetric γ - rigid and γ - soft nuclear system may be addressed by taking the following Hamiltonian into account [20, 22, 23]:

$$H = \chi\hat{T}_r + (1-\chi)\hat{T}_s + V(\beta,\gamma), \qquad \textbf{(1)}$$

where $\hat{T}_r$ is the prolate γ - rigid kinetic energy operator [18] and χ is the rigidity parameter that quantifies the γ -rigidity of the system [23], and

$$\hat{T}_r = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^2\frac{\partial}{\partial\beta} - \frac{Q^2}{3\beta^2}\right]. \qquad \textbf{(2)}$$

Also, $\hat{T}_s$ is the same operator as the typical γ -stable Bohr Hamiltonian in five dimensions

$$\hat{T}_s = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2(\gamma-\frac{2}{3}\pi k)}\right], \qquad \textbf{(3)}$$

where, Q is the angular momentum operator within the intrinsic frame of reference, B is the mass parameter, and $Q_k(k = 1,2,3)$ denotes the operators of its projections. The system provides the $X(5)$ (CPS) for $\chi = 0$, and $X(3)$ model for $\chi = 1$. While, the system generates the $X(4)$ model when the two Hamiltonians are equally mixed, i.e. $\chi = \frac{1}{2}$.

For analytical purposes, it is useful to work with reduced energy and potential:

$$\varepsilon = \frac{2B}{\hbar^2}E, u(\beta,\gamma) = \frac{2B}{\hbar^2}V(\beta,\gamma), \qquad \textbf{(4)}$$

By considering the reduced potential as a sum of two parts

$$u(\beta,\gamma) = u(\beta) + \frac{u(\gamma)}{\beta^2}, \qquad \textbf{(5)}$$

we can produce an exact separation for β and γ variables.

In this work we interested only in β variable which lead us to establish the model [24, 25, 26]. In all cases of the three $X(n)$ models, $(n = 3,4,5)$ the potential $u(\beta)$ is an infinite square well in the form of (6), where the potential at the critical points was considered to be flat [13]. Where β_w is the width of the potential well;

$$u(\beta) = \begin{cases} 0, & \beta \le \beta_w. \\ \infty, & \beta > \beta_w. \end{cases} \qquad \textbf{(6)}$$

By using this potential after separating the variables in the main equation for every considered case of (CPSs), we are left with the Bessel differential equation (7)

$$\left[\frac{\partial^2}{\partial\beta^2} + \frac{1}{\beta}\frac{\partial}{\partial\beta} + (k^2 - \frac{\upsilon^2}{\beta^2})\right]\psi(\beta) = 0. \quad \textbf{(7)}$$

Last equation can be solved by putting $z = k\beta$, then the exact solution is the solution of Bessel equation given by

$$\psi(z) = C_1\, BesselJ_{\upsilon}(z) + C_2\, BesselY_{\upsilon}(z). \quad \textbf{(8)}$$

The boundary condition $\psi(z) = 0$ at $\beta = \beta_w$, determines the eigenvalue in terms of the zeros of the $BesselJ$ function and the scaled energy levels $E(L_s^+)$ of any (CPS) could be calculated from the equation;

$$E(L_s^+) = \frac{BesselJZero[\nu_L, s]^2 - BesselJZero[\nu_0, 1]^2}{BesselJZero[\nu_2, 1]^2 - BesselJZero[\nu_0, 1]^2}, \quad (9)$$

where s indicates the zero's number, and it determines the band of the level. The ground state band takes $s = 1$, while the β_1 band has $s = 2$. Also, $\nu_L = \sqrt{\frac{L(L+1)}{3} + (\frac{3}{2} - \chi)^2}$, where L is the angular momentum quantum number.

All the required values for ν_L, and χ are listed in Table 1. Hence, we can generate the required spectrum for the indicated (CPSs) with overall weighting factor $E(2_1^+)$ needed for the comparison with experimental data for the candidate nuclei.

Table (1): The order of the BesselJ ν_L for different energy levels for the ground state and β_1 bands for the three $X(n)$ models.

	$X(5)$	$X(4)$	$X(3)$
ν_L	$\sqrt{\frac{L(L+1)}{3} + \frac{9}{4}}$	$\sqrt{\frac{L(L+1)}{3} + 1}$	$\sqrt{\frac{L(L+1)}{3} + \frac{1}{4}}$
ν_2	$\sqrt{\frac{17}{4}}$	$\sqrt{3}$	1.5
ν_0	1.5	1	0.5
χ	0	0.5	1

3. SEARCH FOR CANDIDATE X(4) NUCLEI

The candidate nucleus must fulfill each of the requirements listed below in order to be chosen as an $X(4)$ nucleus:

i) It must have the ratio $R_{4/2}$ in the region $2.60 \leq R_{4/2} \leq 2.80$

ii) It has at least seven levels with known spin and parity.

iii) The level energies are well approximated by the predicted energy levels of $X(4)$ model.

iv) The B(E2) ratio for transition between low laying levels $B_{4/2} = \frac{B(E2;4_g^+\to 2_g^+)}{B(E2;2_g^+\to 0_g^+)} = 1.7$.

Out of the whole nuclear landscape, a total of 44 nuclides are selected based on the application of the initial criterion. After implementing the second criterion, a total of 17 potential nuclides remain. To facilitate comparison, Table 2 displays the theoretical energy levels for the three $X(n)$ models alongside the energy levels of the chosen nuclei up to level 20_g^+ and $8_{\beta_1}^+$, as well as the fitted values for these energy levels. The conclusion was then determined by considering the average absolute deviation Δ and the $B_{4/2}$ ratio, where

$$\Delta = \frac{1}{N_L}\sum_i^{N_L}\left|E_i^{ex} - E_i^{Theo}\right|, \quad \textbf{(10)}$$

with E_i^{exp} and E_i^{Theo} being the experimental and theoretical energies respectively, expressed in keV of the $i\ th$ level. Additionally, N_L denotes the number of levels considered in the computations. If $\Delta \leq 1.1$, it will be assumed that the nucleus belongs to a certain model.

4. RESULTS AND DISCUSSION

In order to identify the low-lying collective structure, a comparison of the experimental energy levels of the selected nuclei with the three different sets of theoretical calculations of the $X(n)$ models is provided in Table 2, along with the calculations of the average absolute deviation Δ for the three $X(n)$ models in Table 3. Despite the fact that some of the selected nuclei have a long history as $X(5)$ nuclei, for example, ^{182}Pt, ^{138}Gd [27, 28], ^{126}Ba, ^{130}Ce [29], however, according to Table 2, and Table 3 [30, 31] ^{180}Pt, ^{182}Pt, ^{184}Pt, may be regarded as potential candidates for the $X(4)$ model. In addition, ^{138}Gd and ^{144}Ba, while they satisfy our criteria fairly, their energy levels are constrained and we have not reached a conclusive determination on them. The two nuclei, ^{126}Ba, and ^{110}Ru, exhibit a significant agreement between the actual and theoretical energy levels in the $X(4)$ model. However, it does not meet the requirements in the last criteria. While the g. s-band energy levels of ^{100}Zr exhibit a strong agreement with the $X(4)$ model, there is a notable divergence in the energy level of the β_1 -band. On the other hand, ^{130}Ce may be interpreted by the $X(3)$ model, ^{132}Ce, ^{134}Nd, ^{128}Ba, ^{172}Os, despite good agreement on energy levels with the $X(3)$ model, fail to meet the model's $B_{4/2} = 1.9$ criterion for the $X(3)$ model.

Table (2): Indicate experimental and fit energy levels for candidate nuclei divided by $E(2_1^+)$ and the theoretical values for $X(5)$; $X(4)$; and $X(3)$ models. "Brackets" refers to uncertainties in the value or parity of energy levels.

L^+	$X(3)$	$X(4)$	$X(5)$	^{100}Zr		^{110}Ru		^{112}Ru	
				Exp	*Fit*	*Exp*	*Fit*	*Exp*	*Fit*
2_g^+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
4_g^+	2.44	2.71	2.9	2.66	2.83	2.76	2.68	2.72	2.63
6_g^+	4.23	4.90	5.43	4.99	5.22	5.15	4.82	5.03	4.70
8_g^+	6.35	7.50	8.48	7.94	8.10	8.08	7.36	7.77	7.16
10_g^+	8.78	10.51	12.03	11.41	11.43	11.46	10.30	10.83	9.99
12_g^+	11.52	13.91	16.04	(15.37)	15.19	15.15	13.61	14.05	13.19
14_g^+	14.57	17.70	20.51	(19.78)	19.39	18.07	17.30	17.40	16.74
16_g^+	17.91	21.85	25.44	(24.63)	24.00	21.40	21.35	20.93	20.65
18_g^+	21.56	26.38	30.80	(29.97)	29.03	25.14	25.77	24.63	24.90
20_g^+	25.50	31.28	36.61	(35.82)	34.46	(29.30)	30.54	(28.41)	29.51
$0_{\beta 1}^+$	2.87	4.16	5.65	1.56	5.01	(4.72)	3.98		3.73
$2_{\beta_1}^+$	4.83	6.04	7.45	4.13	6.84	5.80	5.87		5.64
$4_{\beta_1}^+$	7.37	9.01	10.69	6.65	9.98		8.79		8.49
$6_{\beta_1}^+$	10.29	12.55	14.75		13.85		12.26		11.85
$8_{\beta_1}^+$	13.57	16.58	19.44		18.27		16.19		15.65

Continuation of Table (2): Indicate experimental and fit energy levels for candidate nuclei divided by $E(2_1^+)$. "Brackets" refers to uncertainties in the value or parity of energy levels.

L^+	^{126}Ba		^{128}Ba		^{144}Ba		^{130}Ce		^{132}Ce	
	Exp	*Fit*	*Exp*	*Fit*	*Exp*	*Fit*	*Exp*	*Fit*	*Exp*	*Fit*
2_g^+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
4_g^+	2.78	2.63	2.69	2.54	2.65	2.66	2.80	2.47	2.64	2.48
6_g^+	5.20	4.70	4.95	4.46	4.82	4.77	5.22	4.30	4.60	4.32
8_g^+	8.16	7.16	7.71	6.74	7.37	7.28	8.09	6.47	7.16	6.49
10_g^+	11.49	9.98	10.85	9.36	10.25	10.18	11.07	8.96	9.71	8.99
12_g^+	14.64	13.18	14.48	12.31	13.38	13.45	13.05	11.76		11.81
14_g^+	17.26	16.73	16.36	15.60	(16.65)	17.08	15.21	14.88		14.94
16_g^+	20.49	20.63	19.35	19.21	(20.02)	21.08	17.94	18.31		18.39
18_g^+	24.20	24.89	22.66	23.15		25.44	21.21	22.04		22.14
20_g^+	(28.06)	29.49	26.21	27.40		30.15	24.98	26.08		26.19
$0_{\beta 1}^+$	3.84	3.73	3.32	3.26	5.12	3.89	4.04	2.99	3.56	3.01
$2_{\beta_1}^+$	(6.71)	5.63	4.65	5.19	(6.60)	5.78		4.93	4.60	4.96
$4_{\beta_1}^+$		8.48	6.34	7.89		8.67		7.53	5.94	7.56
$6_{\beta_1}^+$		11.84		11.03		12.10		10.51		10.56
$8_{\beta_1}^+$		15.64		14.55		15.98		13.86		13.92

Continuation of Table (2): Indicate experimental and fit energy levels for candidate nuclei divided by $E(2_1^+)$. "Brackets" refers to uncertainties in the value or parity of energy levels.

L^+	^{134}Nd		^{138}Gd		^{158}Er		^{160}Yb		^{172}Os	
	Exp	Fit	Exp	Fit	Exp	Fit	Exp	Fit	Exp	Fit
2_g^+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
4_g^+	2.68	2.36	2.74	2.63	2.74	2.56	2.63	2.27	2.66	2.32
6_g^+	4.83	4.03	4.95	4.69	5.05	4.52	4.72	3.82	4.63	3.95
8_g^+	7.23	6.01	7.47	7.14	7.77	6.84	7.15	5.66	6.70	5.87
10_g^+	9.58	8.27	10.26	9.96	10.79	9.50	9.77	7.76	8.89	8.07
12_g^+	11.84	10.83	13.37	13.15	13.95	12.52	12.18	10.14	11.26	10.55
14_g^+	14.22	13.67	16.81	16.69	17.56	15.86	13.85	12.77	(14.05)	13.31
16_g^+	16.80	16.79	20.61	20.58	(20.95)	19.54	15.84	15.67	(16.79)	16.34
18_g^+	19.64	20.19	24.72	24.83	(24.35)	23.55	18.22	18.82	(19.80)	19.65
20_g^+	22.81	23.87	(29.11)	29.41	(27.73)	27.88	20.95	22.23	(22.98)	23.21
$0_{\beta 1}^+$		2.55		3.71	4.20	3.37	(4.47)	2.25	3.33	2.43
$2_{\beta_1}^+$		4.53		5.62	5.15	5.29	(5.32)	4.25	3.56	4.42
$4_{\beta_1}^+$		6.93		8.46	6.54	8.03	6.55	6.49	5.00	6.75
$6_{\beta_1}^+$		9.66		11.81	8.27	11.21	8.05	9.04	6.81	9.42
$8_{\beta_1}^+$		12.72		15.60	10.51	14.80	9.73	11.88	(9.19)	12.39

Continuation of Table (2): Indicate experimental and fit energy levels for candidate nuclei divided by $E(2_1^+)$. "Brackets" refers to uncertainties in the value or parity of energy levels.

L^+	^{174}Os		^{180}Pt		^{182}Pt		^{184}Pt	
	Exp	Fit	Exp	Fit	Exp	Fit	Exp	Fit
2_g^+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
4_g^+	2.74	2.58	2.68	2.74	2.71	2.68	2.67	2.58
6_g^+	4.90	4.57	4.94	4.98	5.00	4.81	4.90	4.57
8_g^+	7.39	6.94	7.71	7.66	7.78	7.36	7.55	6.92
10_g^+	10.20	9.66	10.93	10.76	10.96	10.29	10.47	9.63
12_g^+	13.33	12.72	14.55	14.25	14.47	13.60	13.52	12.69
14_g^+	16.75	16.13	18.54	18.14	18.27	17.28	16.73	16.09
16_g^+	20.43	19.88	22.87	22.42	22.33	21.33	20.14	19.83
18_g^+	24.35	23.97	27.75	27.08	26.42	25.74	23.74	23.91
20_g^+	28.53	28.38	32.53	32.12	30.51	30.51	27.57	28.31
$0_{\beta 1}^+$	3.44	3.48	3.12	4.37	3.22	3.97	3.02	3.46
$2_{\beta_1}^+$	4.36	5.40	5.62	6.24	5.53	5.86	5.18	5.38
$4_{\beta_1}^+$	6.24	8.17	(8.15)	9.25	8.00	8.78	7.57	8.15
$6_{\beta_1}^+$	8.98	11.41	(10.77)	12.89	10.64	12.25	11.04	11.38
$8_{\beta_1}^+$		15.06		17.02	13.66	16.17		15.02

Table (3): Presents the B(E2) ratio, $R_{4/2}$, the average absolute deviation Δ for the selected nuclides for different models, and their associated fit parameter χ .

				Δ			
Nucleus	$R_{4/2}$	$B_{4/2}$	χ	$X(3)$	$X(4)$	$X(5)$	$X(\chi)$
^{100}Zr	2.66	1.32 ± 0.08	0.2090	3.26	1.75	1.32	1.04
^{110}Ru	2.76	1.30 ± 0.18	0.5657	2.21	0.66	2.03	0.61
^{112}Ru	2.73		0.6561	1.89	0.67	2.65	0.50
^{126}Ba	2.78	1.35 ± 0.07	0.6576	1.86	0.91	2.38	0.72
^{128}Ba	2.69	1.50 ± 0.16	0.8375	1.08	1.44	3.27	0.77
^{144}Ba	2.65	1.89 ± 0.33	0.6003	1.29	0.54	1.80	0.38
^{130}Ce	2.80	1.88 ± 0.24	0.9507	0.87	1.86	3.66	0.90
^{132}Ce	2.64	1.11 ± 0.26	0.9408	0.58	0.83	1.80	0.54
^{134}Nd	2.68	1.45 ± 0.06	1.1408	0.89	2.71	4.86	0.68
^{138}Gd	2.74	1.65 ± 0.04	0.6641	1.72	0.69	2.82	0.18
^{158}Er	2.74	1.44 ± 0.11	0.7961	1.706	1.409	3.378	1.209
^{160}Yb	2.63	1.39 ± 0.10	1.2848	1.52	3.09	5.12	1.09
^{172}Os	2.66	1.50 ± 0.17	1.1977	1.06	2.79	4.80	0.79
^{174}Os	2.74		0.7530	1.37	1.21	3.23	0.66
^{180}Pt	2.68	2.01 ± 0.33	0.4254	2.28	0.71	1.90	0.55
^{182}Pt	2.71	1.68 ± 0.16	0.5686	1.85	0.70	2.45	0.72
^{184}Pt	2.67	1.65 ± 0.09	0.7594	1.13	1.04	3.03	0.44

Employing a suitable methodology in response to the parameter χ is expected to significantly enhance the agreement with experimental observations and provide more suitable experimental implementations. However, the implementation of this action will result in the destruction of the group theoretical framework present in the $X(n)$ cases [23]. All nuclei exhibit predicted behavior, except for ^{182}Pt which fit very well with the $X(4)$ model rather than the fitting approach against the parameter χ , the second nucleus is ^{130}Ce which displays a remarkable fit with the $X(3)$ model. The fitted energy levels and their associated values of Δ are shown in Table 2 and Table 3, respectively, with the $X(\chi)$ column displaying the Δ value.

It is worth mentioning that Budaca et al. [20] have identified four nuclei as prototypes of the X(4) model, namely ^{180}Pt, ^{182}Pt, ^{158}Er, ^{148}Ce. However, it should be noted that ^{148}Ce, with a value $R_{4/2} = 2.86$, falls beyond the scope of our present inquiry. Additionally, ^{158}Er does not meet the model's requirements, since it fails to satisfy the model's criterion with $\Delta > 1.1$.

5. CONCLUSION

The current investigation focuses on the use of the geometrical collective model as a tool of examining the structural characteristics of low-lying excited states in the range of candidate nuclei with $2.60 \leq R_{4/2} \leq 2.80$. In this study, we conduct a comparative analysis of the actual data obtained for the ground state and β_1 bands, and the theoretical data predicted by three $X(n)$ models. The $X(4)$ model demonstrated a high degree of agreement with the low-lying energy level and transition ratios of a restricted set of nuclei. In addition to the research undertaken by Budaca et al. [20], one extra nucleus, ^{184}Pt, has been identified in this work as potential candidate for the $X(4)$ model.

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